RELATIVISTIC KINEMATICS

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0.1 Special Theory of Relativity

0.1.1 Proper time

1. $P_i = (t_i, x_i, y_i, z_i)$ is a particular event in an inertial reference frame K and $P_i' = (t_i', x_i', y_i', z_i')$ is the same event observed in another inertial frame K'.

2. Spatial distance between two events i and j in the frame K is
$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}. \tag{1}$$

3. If a signal is propagating with the speed of light,
$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = c^2(t_i - t_j)^2, \tag{2a}$$
$$(x_i' - x_j')^2 + (y_i' - y_j')^2 + (z_i' - z_j')^2 = c^2(t_i' - t_j')^2, \tag{2b}$$
where the speed of light c is invariant in any inertial reference frame.

4. In the limit $x_j = x_i + dx \to x_i$ and so forth,
$$(cd\tau)^2 \equiv (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{3}$$
$$= (cdt')^2 - (dx')^2 - (dy')^2 - (dz')^2 = 0 = \text{constant}. \tag{4}$$

0.1.2 Time Dilation

5. A particle at rest in an inertial reference frame K' was created at $(t_1', \mathbf{0})$ and then decayed at $(t_2', \mathbf{0})$. In another inertial frame K both events were observed as $(t_1, \boldsymbol{x}_1)$ and $(t_2, \boldsymbol{x}_2)$. The relative velocity between the two frames is $\boldsymbol{v} = \boldsymbol{\beta} c$.

   (a) Show that the particle is moving with the velocity $\boldsymbol{v}$ in frame K.
   $$\boldsymbol{v} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{t_2 - t_1}. \tag{5}$$

   (b) Proper time interval τ is measured at the frame K' where the particle is at rest.
   $$(c\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t)^2\left(1 - \beta^2\right), \tag{6a}$$
   $$\tau = t_2' - t_1' = \frac{t_2 - t_1}{\gamma}, \quad \text{where} \tag{6b}$$
   $$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{6c}$$

   (c) If K' is not an inertial frame, the particle does not move with a constant velocity in an inertial frame K. In this case show that
   $$\tau = t_2' - t_1' = \int_{t_1}^{t_2} \frac{dt}{\gamma(t)}. \tag{7}$$

The lifetime of a particle represents that in its rest frame.

6. $\pi^{\pm}$ and π^0 are the lightest hadrons in nature. $m_{\pi^0} \approx 135$ MeV and $m_{\pi^{\pm}} \approx 140$ MeV. The mean life time of $\pi^{\pm}$ is $(2.6033 \pm 0.0005) \times 10^{-8}$s. If a charged pion is observed at a laboratory with $v = 0.999c$, how long does track of the pion?

$$\begin{aligned}
\tau &= (2.6033 \pm 0.0005) \times 10^{-8}\text{s}, && \text{(8a)}\\
\gamma &= \frac{1}{\sqrt{1-0.999^2}} = 22.3662720421\cdots, && \text{(8b)}\\
\Delta t &= \gamma\tau, && \text{(8c)}\\
\Delta x &= v \times \Delta t = 0.9 \times 299792458\text{m/s} \times 22.3662720421 \times (2.6033 \pm 0.0005) \times 10^{-8}\text{s}\\
&= (157.10 \pm 0.03)\text{m}. && \text{(8d)}
\end{aligned}$$

7. Speed of light c is an invariant quantity. As we have learned, life time of a particle varies depending on the frame. Minimum value of the life time can be measured if the particle is at rest. The life time of a particle usually means the proper life time.

8. A collection of particles with identical mean life time τ is at rest. The number of particles $N(t)$ decreases as time passes by.

$$N(t) = N(t=0)e^{-\Gamma t}. \tag{9}$$

Decay rate is inverse of proper life time

$$\Gamma \equiv \frac{1}{\tau}. \tag{10}$$

The ratio of the number of particles to the initial number measured at proper life time is

$$\frac{N(\tau)}{N(0)} = \frac{1}{e} = 0.367879441171\cdots. \tag{11}$$

9. Decay rate is not Lorentz invariant quantity. Decay rate of a particle approaches the maximum value at the rest frame.

10. The most important assumption of the special theory of relativity is that the speed of light is invariant in any inertial frame. Explain why there is no inertial frame where light is at rest.

0.1.3 Length Contraction

11. Consider a cube with volume L_0^3 placed at an inertial frame K' with a corner matching to the spatial axes. The frame K' is moving with velocity $v\hat{\boldsymbol{x}}$ with respect to the frame K.

12. At $t' = 0$ the origins of the two frames meet each other. At t', the origin of K passes the end of the side. Show that the length of the side along the $x'-$axis in the K' is

$$L_0 = vt'. \tag{12}$$

13. Show that the two events in the K' frame are

$$(0,0,0,0) \quad \text{and} \quad (t', L_0, 0, 0) = \left(\frac{L_0}{v}, L_0, 0, 0\right). \tag{13}$$

14. The origin of K is at rest in the frame K. Show that the above two events measured at the frame K are

$$(0,0,0,0) \quad \text{and} \quad (t, 0, 0, 0) = \left(\frac{L}{v}, 0, 0, 0\right). \tag{14}$$

15. The length L of the side along the $x-$axis measured in the frame K is the speed of the cube v times

$$L = v \times t. \tag{15}$$

Using Lorentz invariance, show that

$$\begin{aligned} (ct)^2 = (ct')^2 - L_0^2 \quad &\rightarrow \quad \frac{L^2}{\beta^2} = \frac{L_0^2}{\beta^2} - L_0^2 \\ &\rightarrow \quad L = \sqrt{1-\beta^2} L_0 = \frac{L_0}{\gamma}. \end{aligned} \tag{16a}$$

16. Show that the volume of the cube measured in the frame K is

$$V = \frac{V_0}{\gamma}. \tag{17}$$

17. There is a box with rest volume V containing N particles. If the box is at rest in the frame K' and the frame is moving with speed v with respect to another inertial frame K, the density of the particle in the box is depending on the frame.

$$\rho[K'] = \frac{N}{V}, \tag{18a}$$

$$\rho[K] = \frac{N}{V/\gamma} = \gamma \rho[K']. \tag{18b}$$

Therefore, the particle density increases with the multiplicative factor γ compared to that measured in the rest frame.

0.2 Lorentz Transformation

1. We would like to use the same dimensions for both time and length. Therefore, we multiply the speed of light to time so that ct has the dimensions of length. And we define the four vector x^μ as
$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z). \tag{19}$$
We call this four-vector by contravariant vector.

2. We assumed
$$(c\tau)^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{20}$$
is invariant for any inertial frame. Especially, the proper time $d\tau$ for the light is vanishing in any case so that the light can travel with the speed of light. Lorentz transformation is the relation between the two four vectors satisfying the invariance condition (20).

3. We use
$$r^2 = \boldsymbol{r}^2 = x^2 + y^2 + z^2, \tag{21}$$
for three-vectors.

4. For four-vectors, we have to define such inner product in a different way because of the sign difference between space and time. A convenient way of writing inner product is defining the four-squared length as
$$x^2 \equiv (ct)^2 - x^2 - y^2 - z^2. \tag{22}$$
If we define covariant vector x_μ as
$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z), \tag{23}$$
we can write
$$x^2 \equiv x_\mu x^\mu. \tag{24}$$
Note that $x_\mu x_\mu$ and $x^\mu x^\mu$ are not Lorentz invariant.
$$x_\mu x_\mu = x^\mu x^\mu = (ct)^2 + x^2 + y^2 + z^2. \leftarrow \text{wrong!} \tag{25}$$

5. Lorentz transformation is then
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{26}$$
keeping the length
$$x'^2 = x^2. \tag{27}$$

6. For a while, we neglect the transverse directions with respect to the relative motion. We consider the case in which the relative motion is along the $x-$axis. By explicit evaluation to see if $c^2t^2 - x^2 = c^2t'^2 - x'^2$, show that the following transform is a Lorentz transformation
$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{28}$$
where α is a real number. Use $\cosh^2\alpha - \sinh^2\alpha = 1$.

7. Show that the inverse transform is

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}. \tag{29}$$

8. Show that the Lorentz transform (28) can be written as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{30}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{31}$$

where $\alpha = \cosh^{-1}\gamma = \tanh^{-1}\beta$

9. Show that K' is moving with a constant velocity $v = \beta c$ along the positive x-axis in K.

10. K' is moving with the velocity $\boldsymbol{\beta}c$ in K. $\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$, $\boldsymbol{\beta}\cdot\boldsymbol{x}_\perp = 0$, and $\boldsymbol{\beta}\cdot\boldsymbol{x}_\parallel = \boldsymbol{\beta}\cdot\boldsymbol{x}$. Show that

$$\boldsymbol{x}' = \boldsymbol{x}'_\parallel + \boldsymbol{x}'_\perp \tag{32a}$$

$$\boldsymbol{x}'_\parallel = \boldsymbol{\beta}\frac{\boldsymbol{\beta}\cdot\boldsymbol{x}'}{\beta^2} \tag{32b}$$

$$\boldsymbol{x}'_\perp = \boldsymbol{x}' - \boldsymbol{\beta}\frac{\boldsymbol{\beta}\cdot\boldsymbol{x}'}{\beta^2} \tag{32c}$$

$$ct' = \gamma\left(ct - \boldsymbol{\beta}\cdot\boldsymbol{x}_\parallel\right) = \gamma\left(ct - \boldsymbol{\beta}\cdot\boldsymbol{x}\right) \tag{32d}$$

$$\boldsymbol{x}'_\parallel = \gamma\left(-\boldsymbol{\beta}ct + \boldsymbol{x}_\parallel\right) = \boldsymbol{\beta}\gamma\left(-ct + \frac{\boldsymbol{\beta}\cdot\boldsymbol{x}}{\beta^2}\right) \tag{32e}$$

$$\boldsymbol{x}'_\perp = \boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{\beta}\frac{\boldsymbol{\beta}\cdot\boldsymbol{x}}{\beta^2} \tag{32f}$$

11. Arranging previous results, show that

$$ct' = \gamma\left(ct - \boldsymbol{\beta}\cdot\boldsymbol{x}\right) \tag{33}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\beta}\left(-\gamma ct + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\cdot\boldsymbol{x}\right) \tag{34}$$

$$= \boldsymbol{x} + \boldsymbol{\beta}\gamma\left(-ct + \frac{\gamma}{1+\gamma}\boldsymbol{\beta}\cdot\boldsymbol{x}\right) \tag{35}$$

12. From Eq. (35), read the matrix representation of the transform matrix $\Lambda^\mu{}_\nu$.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{36a}$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta^1 & -\gamma\beta^2 & -\gamma\beta^3 \\ -\gamma\beta^1 & 1+\frac{(\gamma-1)(\beta^1)^2}{\beta^2} & \frac{(\gamma-1)\beta^1\beta^2}{\beta^2} & \frac{(\gamma-1)\beta^1\beta^3}{\beta^2} \\ -\gamma\beta^2 & \frac{(\gamma-1)\beta^2\beta^1}{\beta^2} & 1+\frac{(\gamma-1)(\beta^2)^2}{\beta^2} & \frac{(\gamma-1)\beta^2\beta^3}{\beta^2} \\ -\gamma\beta^3 & \frac{(\gamma-1)\beta^3\beta^1}{\beta^2} & \frac{(\gamma-1)\beta^3\beta^2}{\beta^2} & 1+\frac{(\gamma-1)(\beta^3)^2}{\beta^2} \end{pmatrix}_{\mu\nu} \tag{36b}$$

From Eq. (35), read the matrix representation of the inverse transform matrix.

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{37a}$$
$$x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu} \tag{37b}$$
$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}(\boldsymbol{\beta} \to -\boldsymbol{\beta}) \tag{37c}$$
$$= \begin{pmatrix} \gamma & \gamma\beta^1 & \gamma\beta^2 & \gamma\beta^3 \\ \gamma\beta^1 & 1+\frac{(\gamma-1)(\beta^1)^2}{\beta^2} & \frac{(\gamma-1)\beta^1\beta^2}{\beta^2} & \frac{(\gamma-1)\beta^1\beta^3}{\beta^2} \\ \gamma\beta^2 & \frac{(\gamma-1)\beta^2\beta^1}{\beta^2} & 1+\frac{(\gamma-1)(\beta^2)^2}{\beta^2} & \frac{(\gamma-1)\beta^2\beta^3}{\beta^2} \\ \gamma\beta^3 & \frac{(\gamma-1)\beta^3\beta^1}{\beta^2} & \frac{(\gamma-1)\beta^3\beta^2}{\beta^2} & 1+\frac{(\gamma-1)(\beta^3)^2}{\beta^2} \end{pmatrix}_{\mu\nu} \tag{37d}$$

Show that

$$\Lambda^{-1}(\boldsymbol{\beta}) = \Lambda(-\boldsymbol{\beta}) = g^T \Lambda(\boldsymbol{\beta}) g. \tag{38}$$

13. Taking the non-relativistic limit, show that in the limit $\beta \to 0$

$$t' = t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}, \tag{39a}$$
$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t. \tag{39b}$$

0.3 Four-vectors and Invariants

0.3.1 Four-vector Notation and Metric Tensor

1. If a vector A satisfies the same transformation condition as that of x, A is a four-vector.

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{40}$$
$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \tag{41}$$

where any repeated indices are summed over 0,1,2, and 3.

2. If both A^μ and B^μ are four-vectors, their inner product is Lorentz invariant.

$$A' \cdot B' = A \cdot B \quad : \quad A'^{\mu} B'_{\mu} = A^{\mu} B_{\mu}. \tag{42}$$

3. Recall **covariant vector** A_μ and **contravariant vector** A^μ:

$$A_\mu \equiv (A^0, -\boldsymbol{A}),\ A^\mu \equiv (A^0, +\boldsymbol{A}) \tag{43}$$

and check the following.

$$A \cdot B \equiv A^0 B^0 - \boldsymbol{A}\cdot\boldsymbol{B} = A_\mu B^\mu = A^\mu B_\mu. \tag{44}$$

4. Metric tensor $g^{\mu\nu} = g_{\mu\nu}$ is useful in the transformation between covariant and contravariant vectors.

$$g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} . \tag{45}$$

Show that

$$A^\mu = g^{\mu\nu} A_\nu, \;\; A_\mu = g_{\mu\nu} A^\nu, \;\; A \cdot B = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu \tag{46}$$

One of any pair of repeated indices is contravariant. And the other is covariant.

5. Show that

$$g^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = g_\mu{}^\nu = g_{\mu\alpha} g^{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu} \tag{47}$$

6. Show that the proof $A' \cdot B' = A \cdot B$ is equivalent to prove

$$\Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta = g_\alpha{}^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\alpha\beta} . \tag{48}$$

0.3.2 Four-Derivative

7. Show that

$$\frac{\partial x^\mu}{\partial x^\nu} = \begin{cases} 1 & \text{if} \quad \mu = \nu, \\ 0 & \text{if} \quad \mu \neq \nu. \end{cases} \tag{49}$$

8. Show that

$$\frac{\partial x^\mu}{\partial x^\mu} = 4 \quad \text{and} \quad \frac{\partial x^\mu}{\partial x^\mu} = 4. \tag{50}$$

Both $\frac{\partial x^\mu}{\partial x^\mu}$ and $\frac{\partial x_\mu}{\partial x_\mu}$ are found to be Lorentz invariant. Therefore,

$$\frac{\partial}{\partial x_\mu} = \text{derivative}^{"\mu"}(\text{contravariant}) \quad \text{and} \quad \frac{\partial}{\partial x^\mu} = \text{derivative}_{"\mu"}(\text{covariant}) \tag{51}$$

9. If you do not trust above argument, you can check it through brutal-force calculation.

   (a) From the definition of Lorentz transformation, show that

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}. \tag{52}$$

(b) From the identity (52), show that

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \tag{53}$$

is a contravariant vector

$$\partial'^\mu = \Lambda^\mu{}_\nu \partial^\nu. \tag{54}$$

(c) With the same method, show that $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ is a covariant vector

$$\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu. \tag{55}$$

10. Show that four-divergence is Lorentz invariant.

$$\partial \cdot J = \partial_\mu J^\mu = \partial^\mu J_\mu = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}, \tag{56}$$

where $J^\mu = (\rho, \boldsymbol{J})$.

11. Show that four-dimensional Laplacian of a Lorentz invariant function is also Lorentz invariant.

$$\partial^2 \equiv \partial \cdot \partial = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial t'^2} - \boldsymbol{\nabla}'^2 = \partial' \cdot \partial'. \tag{57}$$

0.3.3 Derivation of Useful Four-vectors

In this subsection we derive useful four-vectors using known Lorentz covariant four-vectors and Lorentz invariant quantities.

12. We know x^μ is a four-vector.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{58}$$

13.

14. If A^μ is a four-vector and a is a scalar(invariant), aA^μ is a four-vector.

15. Consider a particle at rest in K'

$$ct = \gamma(c\tau + vx'/c) = \gamma c\tau \tag{59a}$$

$$x = \gamma(x' + v\tau) = \gamma\beta c\tau \tag{59b}$$

$$\frac{dx^\mu}{dt} = (c, \boldsymbol{\beta} c) = (c, \boldsymbol{v}) \ \rightarrow \ \frac{dx^\mu}{dt}\frac{dx_\mu}{dt} = c^2(1-\beta^2) \tag{59c}$$

$$\frac{dx'^\mu}{dt'} = \frac{d}{d\tau}(c\tau, \boldsymbol{0}) = (c, \boldsymbol{0}) \ \rightarrow \ \frac{dx'^\mu}{dt'}\frac{dx'_\mu}{dt'} = c^2 \tag{59d}$$

$\frac{dx^\mu}{dt}$ is NOT a four-vector because t is not a scalar.

16. **Four-velocity**
x^μ is a four-vector and proper time τ is a scalar. Show that u^μ

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma(c, \boldsymbol{v}) \;\rightarrow\; u^2 = c^2 \tag{60}$$

is a four-vector. u^μ is the four-velocity of a particle. At rest the four-velocity reduces into

$$u^\mu = (c, \boldsymbol{0}). \tag{61}$$

17. **Four-momentum**
u^μ is a four-vector and the rest mass m_0 is a scalar. Four momentum is a four-vector.

$$p^\mu \equiv m_0 u^\mu = m_0\gamma(c, \boldsymbol{v}), \tag{62a}$$
$$p^2 = m_0^2 c^2. \tag{62b}$$

At rest the four-momentum becomes

$$p^\mu = (m_0 c, \boldsymbol{0}). \tag{63}$$

18. Show that

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) \rightarrow E = mc^2 = m_0\gamma c^2 \tag{64a}$$
$$E = \sqrt{(m_0c^2)^2 + \boldsymbol{p}^2c^2} \tag{64b}$$
$$\gamma = \frac{E}{m_0c^2} \tag{64c}$$
$$\boldsymbol{\beta} = \frac{\boldsymbol{p}c}{E} \tag{64d}$$

19. Show that the Lorentz transform from a frame where a particle is at rest to the frame where the particle is moving with momentum $p^\mu = (E/c, p)$ is

$$\begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} = \frac{1}{mc^2}\begin{pmatrix} E & pc \\ pc & E \end{pmatrix}, \tag{65}$$

where m is the rest mass of the particle.

20. **Velocity addition**
From the Lorentz transformation

$$t' = \gamma\left(t - \frac{\boldsymbol{\beta}\cdot\boldsymbol{x}}{c}\right), \tag{66a}$$
$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\beta}\gamma\left(\frac{\gamma\boldsymbol{\beta}\cdot\boldsymbol{x}}{1+\gamma} - ct\right), \tag{66b}$$

derive the formula for velocity addition

$$\boldsymbol{u}' = \frac{d\boldsymbol{x}'}{dt'} = \frac{\boldsymbol{u} + \boldsymbol{v}\gamma\left[\frac{\gamma \boldsymbol{u}\cdot\boldsymbol{v}}{c^2(1+\gamma)} - 1\right]}{\gamma\left(1 - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}\right)}, \tag{67}$$

where $\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt}$ and $\boldsymbol{v} = \boldsymbol{\beta} c$.

21. From the velocity addition formula, derive the relation for the angle of the trajectory.

$$\tan\theta' = \frac{u'^2}{u'^1} = \frac{u\sin\theta}{\gamma(u\cos\theta - v)}, \tag{68}$$

where $\boldsymbol{u} = (u^1, u^2, 0)$, $\boldsymbol{u}' = (u'^1, u'^2, 0)$, $\boldsymbol{\beta} = \beta(1, 0, 0)$, $\tan\theta = u^2/u^1$

0.4 2-body Kinematics

0.4.1 Center-of-Momentum Frame

1. Let us consider a collision of two particles. In general their momenta are written as

$$p_1 = (E_1, \boldsymbol{p}_1), \quad p_1^2 = m_1^2, \tag{69a}$$
$$p_2 = (E_2, \boldsymbol{p}_2), \quad p_2^2 = m_2^2. \tag{69b}$$

2. Invariant mass of the two particles is defined by

$$M^2 = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2. \tag{70}$$

Evidently, the invariant mass is Lorentz invariant.

3. In a center of momentum frame, expressions are greatly simplified.

$$p_1 = (E_1^*, \boldsymbol{p}_1^*), \tag{71a}$$
$$p_2 = (E_2^*, \boldsymbol{p}_2^*), \ \boldsymbol{p}_1^* + \boldsymbol{p}_2^* = \boldsymbol{0}, \tag{71b}$$
$$M^2 = (p_1 + p_2)^2 = (E_1^* + E_2^*)^2. \tag{71c}$$

Note that the center-of-momentum energy $E_1^* + E_2^*$, total energy in the center-of-momentum frame, is expressed in terms of Lorentz invariant quantity $(p_1 + p_2)^2$. If we know the complete components of the four-momenta for the two particle in any inertial frame, we can calculate the center-of-momentum energy $\sqrt{(p_1 + p_2)^2}$ because the expression is Lorentz invariant.

0.4.2 Fixed-Target or CM

In a fixed target frame, a particle hits the other particle at rest. In the center-of-momentum (CM) frame, the sum of two three-momenta of the colliding particles is vanishing.

4. Let us consider a collision of two particles in any frame K.

   (a) Sum of three-momenta for the two particles are

   $$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2. \tag{72}$$

   If we are in the CM frame, the $\boldsymbol{P}$ vanishes.

   (b) Sum of energies of the two particles are given by

   $$E = E_1 + E_2. \tag{73}$$

   (c) Invariant mass is

   $$M = \sqrt{(p_1 + p_2)^2} = E_1^* + E_2^*. \tag{74}$$

   (d) The velocity and γ factor of the system of particle $P^\mu = (E, \boldsymbol{P})$ are expressed as follows.

   $$\boldsymbol{\beta} = \frac{\boldsymbol{P}}{E} = \frac{\boldsymbol{p}_1 + \boldsymbol{p}_2}{E_1 + E_2}, \tag{75a}$$

   $$\gamma = \frac{E}{M} = \frac{E}{\sqrt{E^2 - \boldsymbol{P}^2}} = \frac{E_1 + E_2}{\sqrt{(E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2}}. \tag{75b}$$

5. Let us define θ by the scattering angle at the fixed-target frame.

   (a) Show that $\cos\theta$ can be expressed in terms of Lorentz invariant quantity.

   $$\cos\theta = \frac{E_1 E_2 - p_1 \cdot p_2}{\sqrt{(E_1^2 - m_1^2)(E_2^2 - m_2^2)}}. \tag{76}$$

   (b) Show that

   $$M^2 = m_1^2 + m_2^2 + 2(E_1 E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2), \tag{77}$$

   where $M^2 = (p_1 + p_2)^2$.

6. Let us consider the case p_1 **is at rest**; $p_1 = (m_1, \mathbf{0})$ and $p_2 = (E_2, \boldsymbol{p}_2)$,

   (a) Express the energy and momentum of the incoming particle (p_2) in terms of Lorentz invariant quantities.

   $$E_2 = \frac{p_1 \cdot p_2}{m_1}, \tag{78a}$$

   $$|\boldsymbol{p}_2| = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{m_1}. \tag{78b}$$

(b) Express the γ factor and velocity β of the incoming particle in terms of Lorentz invariant quantities.

$$\gamma = \frac{E_2}{m_2} = \frac{p_1 \cdot p_2}{m_1 m_2} \tag{79a}$$

$$\beta = \frac{|\boldsymbol{p}_2|}{E_2} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2}. \tag{79b}$$

7. Energy and momentum of each particle in two-body collision can be expressed in terms of Lorentz invariants. Let us first choose the center-of-momentum frame. The four-momenta for the two particles and their sum are $p_1 = (E_1^*, \boldsymbol{p}^*)$, $p_2 = (E_2^*, -\boldsymbol{p}^*)$, and $P = p_1 + p_2 = (M, \mathbf{0})$. Show that

$$M = E_1^* + E_2^*, \tag{80a}$$

$$E_1^* = \frac{P \cdot p_1}{M} = \frac{M^2 + m_1^2 - m_2^2}{2M}, \tag{80b}$$

$$E_2^* = \frac{P \cdot p_2}{M} = \frac{M^2 + m_2^2 - m_1^2}{2M}. \tag{80c}$$

8. The magnitude of the three-momentum of the two particles are same in the center-of-momentum frame. Derive the following formula for the magnitude of the three-momentum in the center-of-momentum frame.

$$|\boldsymbol{p}^*| = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{2M} \tag{81a}$$

$$= \frac{\sqrt{[M^2 - (m_1 + m_2)^2]\,[M^2 - (m_1 - m_2)^2]}}{2M} \tag{81b}$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca) \tag{81c}$$

$$= \left[a - \left(\sqrt{b} + \sqrt{c}\right)^2\right]\left[a - \left(\sqrt{b} - \sqrt{c}\right)^2\right]. \tag{81d}$$

0.4.3 Lorentz transformation to the rest frame of an arbitrary particle

9. p_1^μ is a four-vector. Show that the components of this vector in the frame where $P^\mu = (E, \boldsymbol{P})$ is at rest becomes

$$E_1' = \gamma\,(E_1 - \boldsymbol{\beta} \cdot \boldsymbol{p}_1) \tag{82a}$$

$$\boldsymbol{p}_1' = \boldsymbol{p}_1 + \boldsymbol{\beta}\gamma \left(\frac{\gamma \boldsymbol{\beta} \cdot \boldsymbol{p}_1}{1 + \gamma} - E_1\right) \tag{82b}$$

$$\boldsymbol{\beta} = \frac{\boldsymbol{P}}{E} \tag{82c}$$

$$\gamma = \frac{E}{M}, \; P^2 = M^2 \tag{82d}$$

10. If $p_1 = P$, show that

$$\begin{aligned} E'_1 &= M && (83a)\\ \boldsymbol{p}'_1 &= \boldsymbol{0} && (83b)\end{aligned}$$

11. If $P = p_1 + p_2$, show that p'_1 and p'_2 in the rest frame of the P become

$$\begin{aligned} E'_1 &= \frac{1}{M}\left(EE_1 - \boldsymbol{P}\cdot\boldsymbol{p}_1\right) = \frac{P\cdot p_1}{M} && (84a)\\ E'_2 &= \frac{1}{M}\left(EE_2 - \boldsymbol{P}\cdot\boldsymbol{p}_2\right) = \frac{P\cdot p_2}{M} && (84b)\\ \boldsymbol{p}'_1 &= \boldsymbol{p}_1 + \frac{\boldsymbol{P}}{M}\left(\frac{\boldsymbol{P}\cdot\boldsymbol{p}_1}{E+M} - E_1\right) && (84c)\\ \boldsymbol{p}'_2 &= \boldsymbol{p}_2 + \frac{\boldsymbol{P}}{M}\left(\frac{\boldsymbol{P}\cdot\boldsymbol{p}_2}{E+M} - E_2\right) && (84d)\\ E'_1 + E'_2 &= M && (84e)\\ \boldsymbol{p}'_1 + \boldsymbol{p}'_2 &= \boldsymbol{0} && (84f)\end{aligned}$$

0.5 Lorentz Transformation of integrals

1. There is a function $f(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \cdots, x_n)$ and we want to change the variables into

$$y_k = y_k(x_1, \cdots, x_n), \;\; k = 1, 2, \cdots, n \qquad (85)$$

Show that

$$\begin{aligned} \int_{R_x} f(\boldsymbol{x}) d^n\boldsymbol{x} &= \int_{R_y} g(\boldsymbol{y}) d^n\boldsymbol{y} && (86a)\\ g(\boldsymbol{y}) &= \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} f[\boldsymbol{x}(\boldsymbol{y})], && (86b)\end{aligned}$$

where R_x and R_y are the region of integration with respect to x_k or y_k, respectively.

2. The Jacobian is defined by

$$\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{\partial x_1}{\partial y_n} & \frac{\partial x_2}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}. \qquad (87)$$

3. If $\boldsymbol{\beta} = (\beta, 0, 0)$, show that $p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}. \qquad (88)$$

4. Show that
$$\frac{\partial(p^0p^1p^2p^3)}{\partial(p'^0p'^1p'^2p'^3)} = \text{Det}\Lambda = 1. \tag{89}$$

5. Using the general Lorentz Boost matrix, prove
$$d^4p = d^4p' \leftrightarrow dp^0dp^1dp^2dp^3 = dp'^0dp'^1dp'^2dp'^3. \tag{90}$$
This is equivalent to
$$\frac{\partial(p^0p^1p^2p^3)}{\partial(p'^0p'^1p'^2p'^3)} = \text{Det}\Lambda = 1. \tag{91}$$
Therefore, d^4p is Lorentz invariant.

6. Show that
$$\delta[f(x)] = \sum \frac{\delta(x-x_i)}{\left|\frac{df}{dx}(x_i)\right|}, \tag{92a}$$
where $f(x)$ has a simple pole at $x = x_i$.

7. Show that
$$\delta(x^2 - a^2) = \frac{\delta(x+|a|) + \delta(x-|a|)}{2|a|}. \tag{93}$$

8. Show that $\delta(p^2 - m^2)d^4p$ is Lorentz invariant. $E = \sqrt{m^2 + \boldsymbol{p}^2}$
$$\begin{aligned}\delta(p^2 - m^2)d^4p &= \delta\left((p^0 - E)(p^0 + E)\right) dp^0dp^1dp^2dp^3 \\ &= \frac{\delta(p^0 - E) + \delta(p^0 + E)}{2E} dp^0dp^1dp^2dp^3.\end{aligned} \tag{94a}$$

9. Show that
$$\theta(p^0)\delta(p^2 - m^2)d^4p = \frac{d^3\boldsymbol{p}}{2E}. \tag{95}$$
is Lorentz invariant.

10. Explain the Lorentz invariance of d^4p and $d^3\boldsymbol{p}/(2E)$ using only time dilation and length contraction.

0.6 Kinematics of 2-2 Scattering

0.6.1 Independent Kinematic Variables

1. Transition amplitude of four particles with momenta p_1, p_2, p_3, and p_4 is a function of the four four-momenta
$$T_{fi} = T(p_1, p_2, p_3, p_4), \tag{96}$$

where we ignore any other variables like spin, isospin, so on. The function is a Lorentz scalar. We want to know how many independent Lorentz scalars made of the scalar products of the four four-momenta exist.

2. Show that there are 10 scalar products made of the four four-vectors.

$$4(\text{from }\ p_i^2) + \frac{4\cdot 3}{2}(\text{from }\ p_i\cdot p_j,\ i\neq j) = 10. \tag{97}$$

3. Among the 10 scalar products

$$p_1^2 = m_1^2,\ p_2^2 = m_2^2,\ p_3^2 = m_3^2,\ p_4^2 = m_4^2 \tag{98}$$

are known from on-shell conditions so that we have at most $10-4=6$ independent scalar products.

4. In addition, there are four more conditions from energy-momentum conservation

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{99}$$

Finally we end up with only two independent scalar products.

0.6.2 Lab(fixed-target) system

5.

$$\begin{aligned} p_1(m_1) + p_2(m_2) &\to p_3(m_3) + p_4(m_4) && \text{(100a)}\\ p_1 &= (E_1, \boldsymbol{p}_1),\ p_2 = (m_2, \mathbf{0}) && \text{(100b)}\\ p_3 &= (E_3, \boldsymbol{p}_3),\ p_4 = (E_4, \boldsymbol{p}_4) && \text{(100c)}\end{aligned}$$

6. Energy-momentum conservation requires

$$E_1 + m_2 = E_3 + E_4, \quad \boldsymbol{p}_1 = \boldsymbol{p}_3 + \boldsymbol{p}_4. \tag{101}$$

7. For $i=1,3,4$, show that

$$\begin{aligned} E_i &= \frac{p_2\cdot p_i}{m_2} && \text{(102a)}\\ |\boldsymbol{p}_i| &= \frac{\sqrt{(p_2\cdot p_i)^2 - m_2^2 m_i^2}}{m_2} && \text{(102b)}\end{aligned}$$

8. If the scattering angle $\theta_i (i=3,4)$ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_i$, show that

$$\cos\theta_i = \frac{p_1\cdot p_2 p_i\cdot p_2 - p_1\cdot p_i m_2^2}{\sqrt{[(p_1\cdot p_2)^2 - m_1^2 m_2^2]\,[(p_i\cdot p_2)^2 - m_i^2 m_2^2]}}. \tag{103a}$$

0.6.3 Center-of-Momentum(CM) system

9.

$$\begin{aligned} p_1(m_1)+p_2(m_2) &\to p_3(m_3)+p_4(m_4), &(104a)\\ p_1 &= (E_1,\boldsymbol{p}_1),\ p_2=(E_2,\boldsymbol{p}_2), &(104b)\\ p_3 &= (E_3,\boldsymbol{p}_3),\ p_4=(E_4,\boldsymbol{p}_4), &(104c)\\ P &= p_1+p_2=p_3+p_4=(E,\boldsymbol{P}). &(104d)\end{aligned}$$

In the CM system, energy-momentum conservation requires

$$E=M=E_1+E_2=E_3+E_4,\quad \boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}_3+\boldsymbol{p}_4=\boldsymbol{0}. \qquad (105)$$

10. Show that $\sqrt{s}$ is the CM energy E;

$$\begin{aligned} s &\equiv (p_1+p_2)^2=(p_3+p_4)^2=E^2, &(106a)\\ p_1\cdot p_2 &= \frac{1}{2}\left(s-m_1^2-m_2^2\right),\quad p_3\cdot p_4=\frac{1}{2}\left(s-m_3^2-m_4^2\right), &(106b)\end{aligned}$$

11. Show that

$$\begin{aligned} p_1\cdot p_3 &= -\frac{1}{2}\left(t-m_1^2-m_3^2\right),\quad p_2\cdot p_4=-\frac{1}{2}\left(t-m_2^2-m_4^2\right), &(107a)\\ p_1\cdot p_4 &= -\frac{1}{2}\left(t-m_1^2-m_4^2\right),\quad p_2\cdot p_3=-\frac{1}{2}\left(t-m_2^2-m_3^2\right), &(107b)\\ t &\equiv (p_1-p_3)^2=(p_2-p_4)^2,\quad u\equiv(p_1-p_4)^2=(p_2-p_3)^2, &(107c)\\ s+t+u &= m_1^2+m_2^2+m_3^2+m_4^2. &(107d)\end{aligned}$$

12. Show that in the CM system

$$\begin{aligned} E_1 &= \frac{p_1\cdot P}{M}=\frac{s+m_1^2-m_2^2}{2\sqrt{s}},\quad E_2=\frac{p_2\cdot P}{M}=\frac{s+m_2^2-m_1^2}{2\sqrt{s}} &(108a)\\ E_3 &= \frac{p_3\cdot P}{M}=\frac{s+m_3^2-m_4^2}{2\sqrt{s}},\quad E_4=\frac{p_4\cdot P}{M}=\frac{s+m_4^2-m_3^2}{2\sqrt{s}}. &(108b)\end{aligned}$$

13. Show that in the CM system

$$\begin{aligned} |\boldsymbol{p}_1| &= |\boldsymbol{p}_2|=\frac{\sqrt{(p_1\cdot P)^2-m_1^2 s}}{\sqrt{s}}=\frac{\sqrt{(p_2\cdot P)^2-m_2^2 s}}{\sqrt{s}}\\ &= \frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}}=\frac{\sqrt{[s-(m_1+m_2)^2]\,[s-(m_1-m_2)^2]}}{2\sqrt{s}} &(109a)\\ |\boldsymbol{p}_3| &= |\boldsymbol{p}_4|=\frac{\sqrt{(p_3\cdot P)^2-m_3^2 s}}{\sqrt{s}}=\frac{\sqrt{(p_4\cdot P)^2-m_4^2 s}}{\sqrt{s}}\\ &= \frac{\lambda^{1/2}(s,m_3^2,m_4^2)}{2\sqrt{s}}=\frac{\sqrt{[s-(m_3+m_4)^2]\,[s-(m_3-m_4)^2]}}{2\sqrt{s}} &(109b)\end{aligned}$$

14. Show that

$$\frac{1}{2}(1-\cos\theta) = \sin^2\frac{\theta}{2}, \tag{110a}$$
$$\frac{1}{2}(1+\cos\theta) = \cos^2\frac{\theta}{2}. \tag{110b}$$

15. If the scattering angle θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ in the CM frame, show that

$$t = (p_1-p_3)^2 = (E_1-E_3)^2 - (\boldsymbol{p}_1-\boldsymbol{p}_3)^2 \tag{111a}$$
$$= (E_1-E_3)^2 - (|\boldsymbol{p}_1|-|\boldsymbol{p}_3|)^2 - 4|\boldsymbol{p}_1||\boldsymbol{p}_3|\sin^2\frac{\theta}{2}, \tag{111b}$$

$$t_{\min} < t < t_{\max}, \tag{111c}$$
$$t_{\max} = (E_1-E_3)^2 - (|\boldsymbol{p}_1|-|\boldsymbol{p}_3|)^2, \tag{111d}$$
$$t_{\min} = (E_1-E_3)^2 - (|\boldsymbol{p}_1|+|\boldsymbol{p}_3|)^2, \tag{111e}$$
$$E_1-E_3 = \frac{m_1^2-m_2^2-m_3^2+m_4^2}{2\sqrt{s}}. \tag{111f}$$

16. If the scattering angle θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$, show that

$$\cos\theta = \frac{p_1\cdot P p_3\cdot P - p_1\cdot p_3 s}{\sqrt{[(p_1\cdot P)^2-m_1^2 s]\,[(p_3\cdot P)^2-m_3^2 s]}}. \tag{112}$$

0.7 Phase Space

1. Show that

$$\int e^{ipx}dx = 2\pi\delta(p) \tag{113}$$
$$\int e^{ip\cdot x}d^4x = (2\pi)^4\delta^4(p) = (2\pi)^4\delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \tag{114}$$

2. Consider a scattering

$$k_1+k_2 \to p_1(m_1)+p_2(m_2)+\cdots+p_n(m_n). \tag{115}$$

Show that the energy-momentum-conservation delta function

$$\int e^{i(p_1+p_2+\cdots+p_n)\cdot x}e^{-i(k_1+k_2)\cdot x}d^4x = (2\pi)^4\delta^4\left[\sum_{i=1}^{n}p_i-(k_1+k_2)\right] \tag{116}$$

is Lorentz invariant.

3. Show that the phase-space element

$$\delta^4\left[\sum_{i=1}^{n} p_i - (k_1+k_2)\right]\prod_{i=1}^{n}\delta(p_i^2 - m_i^2)d^4p_i \tag{117}$$

is Lorentz invariant.

4. Show that

$$\begin{aligned} & \delta^4\left[\sum_{i=1}^{n} p_i - (k_1+k_2)\right]\prod_{i=1}^{n}\delta(p_i^2 - m_i^2)d^4p_i && \text{(118a)} \\ = & \ \delta\left(\sum_{i=1}^{n} E_i - E_1(k_1) - E_2(k_2)\right)\delta^3\left[\sum_{i=1}^{n}\boldsymbol{p}_i - \boldsymbol{k}_1 - \boldsymbol{k}_2\right]\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{2E_i} && \text{(118b)} \end{aligned}$$

is Lorentz invariant, where $p_i^0 = E_i = \sqrt{m_i^2 + \boldsymbol{p}_i^2}$ and $E_i(k_i) = \sqrt{M_i^2 + \boldsymbol{k}_i^2}$.

5. Show that

$$\begin{aligned} d^3\boldsymbol{p} &= |\boldsymbol{p}|^2 d|\boldsymbol{p}|d\Omega = \frac{1}{2}|\boldsymbol{p}|d\boldsymbol{p}^2 d\Omega, && \text{(119a)} \\ d\Omega &= \sin\theta d\theta d\phi, \quad 0 < \theta < \pi, \ 0 < \phi < 2\pi && \text{(119b)} \\ &= d\cos\theta d\phi, \quad -1 < \cos\theta < 1. && \text{(119c)} \end{aligned}$$

6. If the angular dependence of the integrand is independent of the particular direction of $\boldsymbol{p}_1$, show that

$$d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 = 4\pi|\boldsymbol{p}_1|^2 d|\boldsymbol{p}_1||\boldsymbol{p}_2|^2 d|\boldsymbol{p}_2| d\cos\theta_{12} d\phi_{12}, \tag{120a}$$

where $\Omega_{12}(\theta_{12}, \phi_{12})$ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.

7. **n-body phase space element**

$$d\Phi_n = (2\pi)^4\delta^4\left[\sum_{i=1}^{n} p_i - (k_1+k_2)\right]\prod_{i=1}^{n}\frac{d^3\boldsymbol{p}_i}{2E_i(2\pi)^3} \tag{121}$$

8. **2-body phase space element**

$$\begin{aligned} d\Phi_2 &= (2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)\frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2}{4E_1E_2(2\pi)^6} && \text{(122a)} \\ &= (2\pi)^{-2}\delta(E_1 + E_2 - \sqrt{s})\frac{d^3\boldsymbol{p}_1}{4E_1E_2} && \text{(122b)} \end{aligned}$$

9. Show that in the CM frame

$$
\begin{aligned}
E_1 &= \sqrt{m_1^2 + \boldsymbol{p}^2}, \quad E_2 = \sqrt{m_2^2 + \boldsymbol{p}^2} && \text{(123a)}\\
\sqrt{s} &= E_1 + E_2 && \text{(123b)}\\
\boldsymbol{p}_1 &= -\boldsymbol{p}_2 = \boldsymbol{p}, \ |\boldsymbol{p}| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} && \text{(123c)}\\
\frac{\partial E_1}{\partial |\boldsymbol{p}|} &= \frac{|\boldsymbol{p}|}{E_1}, \quad \frac{\partial E_2}{\partial |\boldsymbol{p}|} = \frac{|\boldsymbol{p}|}{E_2}, && \text{(123d)}\\
\delta(E_1 + E_2 - \sqrt{s}) &= \frac{\delta(|\boldsymbol{p}_1| - |\boldsymbol{p}|)}{\frac{|\boldsymbol{p}|}{E_1} + \frac{|\boldsymbol{p}|}{E_2}} && \text{(123e)}\\
&= \frac{E_1 E_2}{|\boldsymbol{p}|\sqrt{s}}\delta(|\boldsymbol{p}_1| - |\boldsymbol{p}|) && \text{(123f)}
\end{aligned}
$$

10. Therefore

$$
\begin{aligned}
d\Phi_2 &= \frac{1}{(2\pi)^2} \cdot \frac{|\boldsymbol{p}|}{4\sqrt{s}} d\Omega && \text{(124a)}\\
|\boldsymbol{p}| &= \frac{\lambda^{1/2}(s, p_1^2, p_2^2)}{2\sqrt{s}} && \text{(124b)}
\end{aligned}
$$

this is valid in any inertial frame because of the Lorentz invariance.
In case there is no angular dependence in the integrand(matrix element),

$$\int d\Phi_2 = \frac{|\boldsymbol{p}|}{4\pi\sqrt{s}} \tag{125}$$

If $p_1^2 = p_2^2 = 0$,

$$\int d\Phi_2 = \frac{1}{8\pi} \tag{126}$$

11. **3-body phase space element** Show that

$$
\begin{aligned}
d\Phi_3 &= (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - k_1 - k_2) \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 d^3\boldsymbol{p}_3}{8E_1E_2E_3(2\pi)^9} && \text{(127a)}\\
&= (2\pi)^{-5} \delta\left(E_1 + E_2 + E_3 - \sqrt{s}\right) \frac{d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2}{8E_1E_2E_3} && \text{(127b)}
\end{aligned}
$$

In the $k_1 + k_2$ CM frame show that

$$
\begin{aligned}
E_1 &= \sqrt{m_1^2 + \boldsymbol{p}_1^2}, \quad E_2 = \sqrt{m_1^2 + \boldsymbol{p}_2^2} && \text{(128a)}\\
E_3 &= \sqrt{m_3^2 + \boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 + 2|\boldsymbol{p}_1||\boldsymbol{p}_2|\cos\theta_{12}} && \text{(128b)}
\end{aligned}
$$

12. Show that

$$\begin{aligned} \frac{\partial E_1}{\partial|\boldsymbol{p}_1|} &= \frac{|\boldsymbol{p}_1|}{E_1}, \quad \frac{\partial E_2}{\partial|\boldsymbol{p}_2|} = \frac{|\boldsymbol{p}_2|}{E_2} && \text{(129a)} \\ \frac{\partial E_3}{\partial\cos\theta_{12}} &= \frac{|\boldsymbol{p}_1||\boldsymbol{p}_2|}{E_3} && \text{(129b)} \\ d|\boldsymbol{p}_1|d|\boldsymbol{p}_1|d\cos\theta_{12} &= \frac{E_1E_2E_3}{\boldsymbol{p}_1^2\boldsymbol{p}_2^2}dE_1dE_2dE_3 && \text{(129c)} \end{aligned}$$

$$\begin{aligned} d\Phi_3 &= (2\pi)^{-5}\delta\left(E_1+E_2+E_3-\sqrt{s}\right)\times 4\pi d\cos\theta_{12}d\phi_{12} && \text{(130a)} \\ &\times \frac{\boldsymbol{p}_1^2 d|\boldsymbol{p}_1|\boldsymbol{p}_2^2 d|\boldsymbol{p}_2|}{8E_1E_2E_3} && \text{(130b)} \\ &= (2\pi)^{-5}dE_1dE_2\frac{\pi d\phi_{12}}{2}. && \text{(130c)} \end{aligned}$$